Convex Combinations of Stable Matrices

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Abstract: *New sufficient conditions are introduced for Hurwitz and Schur stability of a related problem to interval matrices which is convex combinations of stable matrices.*

We would like to discuss a related problem, namely, stability of convex combinations of stable matrices. The motivation for this study is to derive conditions which allow for generating from stable interval matrices other stable ones.

SOH (1990) has shown that the convex combina on of symmetric matrices A and B is stable if and only if A and B are stable. In general, it is well known that convex combination of two stable matrices is not necessarily stable.

Let A and B be two $n \times n$ matrices with positive diagonals and non- positive off diagonal elements. If

$$a_{ii}^{-1} a_{ij} \geq b_{ii}^{-1} b_{ij} \qquad i,j = 1,2,\ldots,n$$

Then A is said to proportionally dominate B row wise. Column wise proportional domination is defined similarly. It was shown by Sezer and Siljak (1994) that if $-A$ and $-B$ are two

matrices with one proportionally dominating the other, then the convex combination is stable. We show by the following discussion that the proportional domination requirement is a sever one.

Theorem 1

Let $A_k = [{}_k a_{ii}]$ be $n \times n$ Hurwitz stable matrices with negative diagonal elements and non-negative off diagonal elements with $k = 0,1,\ldots,N.$ Let

$$T = [t_{ij}], \qquad t_{ij} = \max_{k\,{}_k a_{ii}} \square \quad i,j = 1,2,\ldots \square$$

If T is Hurwitz stable, then the convex combination $\sum_{k}^{N} \alpha_k A_k, \sum_{k=0}^{N} \alpha_k = 1, \qquad \alpha_k \geq 0$ is Hurwitz stable.

Proof:

Assume that T is Hurwitz stable and notice that

$$ {}_k a_{ii}^{-1}\ {}_k a_{ij} \geq t_{ii}^{-1} t_{ij} \qquad \forall\, k = 0,1,\dots,\square $$

Which implies that $-A_k$ is proportionally dominating $-T$ for every $k = 0,\dots,N.$ According to Theorem 4 by Sezer and Siljak, $\sum_{k}^{N} \alpha_k A_k$ is Hurwitz stable.

Q.E.D

Example:

Consider the the two matrices $A_0 = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -2 & 1.5 \\ 0.9 & -3 \end{bmatrix}$. Show that the convex combinations $\alpha A_0 + (1-\alpha)A_1$ is Hurwitz stable for $\alpha \geq 0.$

Choose $T = \begin{bmatrix} -2 & 1.5 \\ 1 & -3 \end{bmatrix}$. According to the theorem, $\alpha A_0 + (1-\alpha)A_1$ is Hurwtiz stable.

Note that neither $-A_0$ proportionally dominates $-A_1$ nor $-A_1$ proportionally dominates $-A_0$. Therefore, the result by Sezer and Siljak is inconclusive about the Hurwitz stability of the convex combination $\alpha A_0 + (1-\alpha)A_1, \ \alpha \in [0,1]$ while our result concludes its Hurwitz stability.

In fact Theorem 1 says that the polytope resul ng from the convex combina ons of the matrices $A_k, k = 0,\dots,N$ is stable if it is contained in the Hurwtiz stable hyperrectangle in the parameter space of the off diagonal elements of the test matrix T and which is open toward $-\infty$ in the parameter space of the diagonal elements of $T.$

In the following theorem, we present another sufficient condition for the Hurwitz stability of a convex combination of a set of matrices. We will show by an example, that this condition improves the one presented in Theorem 1 above.

Theorem 2:

Let $A_k, k = 0,\dots N$ be $n \times n$ Hurwtiz stable matrices with negative diagonal elements and non-negative off diagonal elements. If there exists a non-negative positive definite matrix R such that $A_k R + R A_k^T$ is negative definite for $k = 0, \ldots N$, then the convex combination $\sum_{k=0}^{N} \alpha_k A_k$, $\alpha \geq 0$, $\sum_{k=0}^{N} \alpha_k = 1$ is also Hurwitz stable.

Proof:

Let R be such that $A_k R + R A_k^T = Q_k$, $k = 0, \ldots, N$ where Q_k are negative definite, then

$$\left(\sum_{k=0}^{N} \alpha_k A_k\right) R + R\left(\sum_{k=0}^{N} \alpha_k A_k^T\right) = \sum_{k=0}^{N} \alpha_k Q_k$$

Notice that $\sum_{k=0}^{N} \alpha_k Q_k$ is negative definite because it is the convex combination of negative definite matrices (SOH 1990). This implies that $\sum_{k=0}^{N} \alpha_k A_k$ is Hurwitz stable.

Q.E.D

Example:

Consider the the two matrices $A_0 = \begin{bmatrix} -2 & 1 \\ 2 & -[illegible] \end{bmatrix}$, $A_1 = \begin{bmatrix} -2 & 3 \\ 0.9 & -3 \end{bmatrix}$. Show that the convex combinations $\alpha A_0 + (1 - \alpha) A_1$ is Hurwitz stable.

For $R = I$, we have $A_k R + R A_k^T$ are negative definite for $k = 0,1$ which implies that

is stable. Using Theorem 1, the test matrix $T = \begin{bmatrix} -2 & 3 \\ 0.9 & -3 \end{bmatrix}$ is unstable and consequently is inconclusive about the Hurwitz stability of the convex combinations of A_0 and A_1.

One issue that need further study and investigation is the existence and construction of the matrix R in Theorem 2.

Graph or Image,,

In the above two theorems, we restricted treatment to matrices with negative diagonals and non-negative off diagonal matrices. In the following theorem we eliminate this restriction.

Theorem 3:

Let $A_k, k = 0, \dots N$ be any $n \times n$ Hurwitz stable matrices. Assume that the matrix

$$M = \begin{bmatrix} A_0 & \eta I \\ -\eta I & -A_0^T \end{bmatrix}, \eta = \max_{k=0,\dots,N} \|A_k - A_0\|_\infty$$

Has no imaginary Eigenvalues, then the convex combinations $\sum_{k=0}^{N} \alpha_k A_k, \alpha \geq 0, \quad \sum_{k=0}^{N} \alpha_k = 1$ is Hurwitz stable.

Proof:

Note that $\sum_{k=0}^{N} \alpha_k A_k = A_0 + \sum_{k=0}^{N} \alpha_k (A_k - A_0), \quad \sum_{k=0}^{N} \alpha_k = 1$. This matrix is Hurwitz stable

if $\left\| (sI - A_0)^{-1} \sum_{k=0}^{N} (A_k - A_0) \right\|_\infty < 1$ for . To show this we have

$$\left\| (sI - A_0)^{-1} \sum_{k=0}^{N} (A_k - A_0) \right\|_\infty \leq \sum_{k=0}^{N} \alpha_k \|(sI - A_0)^{-1}\|_\infty \|A_k - A_0\|_\infty$$

$$\leq \sum_{k=0}^{N} \alpha_k \|(sI - A_0)^{-1}\|_\infty \, \eta$$

$$= \sum_{k=0}^{N} \alpha_k \|(sI - A_0)^{-1}\|_\infty \, \eta$$

$$= \|(sI - A_0)^{-1}\|_\infty \eta$$

$$= \|(sI - A_0)^{-1}\eta\|_\infty$$

But since the matrix M has no imaginary Eigenvalues, then by Theorem( ) we have $\|(sI - A_0)^{-1}\eta\|_\infty < 1$. This completes the proof.

Q.E.D

Example:

$A_0 = \begin{bmatrix} -3.1 & -2.1 \\ 0.9 & = 0.1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -3.1 & -1.6 \\ 0.95 & -0.1 \end{bmatrix}$, $A_2 = \begin{bmatrix} -3.1 & -1.9 \\ 1.2 & -0.1 \end{bmatrix}$

$\eta = 0.5$, $M = \begin{bmatrix} A_0 & 0.5I \\ = 0.5I & -A_0^T \end{bmatrix}$

The matrix M has no imaginary Eigenvalues. Therefore $\sum_{k=0}^{N} \alpha_k A_k$ is Hurwitz stable.

Image>>

Theorem 4:

Let $A_k = [{}_k a_{ij}]$ be $n \times n$ Shur stable matrices such that $A_k \geq 0$ (i.e. all elements are non-negative) for $k = 0,1,\ldots,N$. Let $T = [t_{ij}]$ such that $t_{ij} = max\{{}_k a_{ij}\}$ for $i,j = 1,2,\ldots,n$. If

T is Schur stable, then the convex combination $\sum_{k=0}^{N} \alpha_k A_k, \alpha \geq 0, \quad \sum_{k=0}^{N} \alpha_k = 1$ is Schur stable.

Proof:

Notice that the matrices $I - A_k$ for $k = 0,1,\dots,N$ proportionally dominate $I - T$ row wise. I.e.

$$(1 - {}_k a_{ii})^{-1} {}_k a_{ij} \geq [(1 - t]_{ii})^{-1} t_{ij} \qquad \forall\, k = 0,1,\dots,N$$

Using Theorem 3 in Sezer and Siljak we conclude that $\sum_{k=0}^{N} \alpha_k A_k$ is Schur stable.

Q.E.D

Theorem 5:

Let $A_k = [{}_k a_{ij}]$ be $n \times n$ Shur stable matrices such that $A_k \geq \mathbf{0}$ (i.e. all elements are non-negative) for $k = 0,1,\dots,N$. If there exists a non-negative positive definite matrix R such that

$(A_k - I)R + R(A_k - I)^T$ is negative definite for all then the convex combination $\sum_{k=0}^{N} \alpha_k A_k, \alpha \geq 0, \quad \sum_{k=0}^{N} \alpha_k = 1$ is Schur stable.

Proof:

Let R be such that $(A_k - I)R + R(A_k - I)^T = Q_k$ where Q_k are negative definite for all $k = 0,1,\dots,N$.

$$\sum_{k=0}^{N} \alpha_k (A_k - I)R + R \sum_{k=0}^{N} \alpha_k (A_k - I)^T = \sum_{k=0}^{N} \alpha_k Q_k$$

Notice that $\sum_{k=0}^{N} \alpha_k Q_k$ is nega ve definite (Soh 1990) and consequently $\sum_{k=0}^{N} \alpha_k (A_k - I)^{\square} = \sum_{k=0}^{N} \alpha_k A_k - I^{\square}$ is Hurwitz stable. From positive matrices properties, this implies that $\sum_{k=0}^{N} \alpha_k A_k^{\square} \geq \mathbf{0}$ must be Schur stable.

Q.E.D

In the above two theorem, we restricted our discussion to matrices with non-negative elements. In the following theorem we eliminate this restriction.

Theorem 6:

Let $A_k, k = 0, \ldots N$ be any $n \times n$ Schur stable matrices. Assume that the matrix

$$M = \begin{bmatrix} A_0 - \eta^2 A_0^{-T} & A_0^{-T} \\ -\eta^2 A_0^{-T} & -A_0^{T} \end{bmatrix}, \eta = \max_{k=0,\ldots,N} \|A_k - A_0\|_\infty$$

Has no imaginary Eigenvalues, then the convex combinations $\sum_{k=0}^{N} \alpha_k A_k, \alpha \geq 0, \quad \sum_{k=0}^{N} \alpha_k = 1$ is Hurwitz stable.

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